Verified Compilation of IMP to Linear IMP

Initial Bachelor Seminar Talk

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Content

Previous Work

Glynn Winskel
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MIT Press, 1993

Benjamin C. Pierce, Chris Casinghino, Marco Gaboardi, Michael Greenberg, Catalin Hritcu, Vilhelm Sjoberg, and Brent Yorgey.
Software Foundations
Electronic textbook, 2015

Tobias Nipkow, Gerwin Klein
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A First-Order Functional Intermediate Language for Verified Compilers
CoRR, abs/1503.08665, 2015

Motivation

Example

$$
\begin{array}{l}
\textbf{if } x < 0 \textbf{ then } x ::= -x \textbf{ else SKIP} \; ; \\
\textbf{while } n > 1 \textbf{ do} \\
\qquad n ::= n - 1 \; ; \\
\qquad x ::= x \cdot x
\end{array}
$$

$$c, d \;::=\; x ::= e \;\mid\; c \,;\, d \;\mid\; \textbf{if } b \textbf{ then } c \textbf{ else } d \;\mid\; \textbf{while } b \textbf{ do } c \;\mid\; \textbf{SKIP}$$

$$\text{where } e \in \text{AExp},\; b \in \text{BExp}$$

- The (terminating) execution of the program changes the state

IMP

Example

if $x < 0$ **then** $x ::= -x$ **else SKIP** ;
while $n > 0$ **do**
$\quad n ::= n - 1$;
$\quad x ::= x \cdot x$

Example

if $b_{x<0}$ **then** $a_{(x ::= -x)}$ **else** $a_{\textbf{SKIP}}$;
while $b_{n>0}$ **do**
$\quad a_{(n ::= n-1)}$;
$\quad a_{(x ::= x \cdot x)}$

IMP: $c, d \ ::= \ a \ | \ c \,;\, d \ | \ \textbf{if } b \textbf{ then } c \textbf{ else } d \ | \ \textbf{while } b \textbf{ do } c$

$$\Sigma = \mathcal{V} \to \mathbb{V} \qquad a : \Sigma \to \Sigma \qquad b : \Sigma \to \mathbb{B}$$

- **SKIP** can be treated as an action that leaves the state unchanged
- Neither arithmetic nor boolean expressions have to be specified

Big-Step Semantics

- relates initial and final state of an execution
- BS $c\ \sigma\ \tau$:= the execution of c in state σ terminates in state τ

$$\frac{a\,\sigma = \tau}{\mathrm{BS}\ a\ \sigma\ \tau} \qquad \frac{\mathrm{BS}\ c\ \sigma_1\ \sigma_2 \quad \mathrm{BS}\ d\ \sigma_2\ \sigma_3}{\mathrm{BS}\ (c\ ;\ d)\ \sigma_1\ \sigma_3} \qquad \frac{b\,\sigma = \mathit{true} \quad \mathrm{BS}\ c\ \sigma\ \tau}{\mathrm{BS}\ (\textbf{if}\ b\ \textbf{then}\ c\ \textbf{else}\ d)\ \sigma\ \tau}$$

$$\frac{b\,\sigma = \mathit{false} \quad \mathrm{BS}\ d\ \sigma\ \tau}{\mathrm{BS}\ (\textbf{if}\ b\ \textbf{then}\ c\ \textbf{else}\ d)\ \sigma\ \tau} \qquad \frac{b\,\sigma = \mathit{false}}{\mathrm{BS}\ (\textbf{while}\ b\ \textbf{do}\ c)\ \sigma\ \sigma}$$

$$\frac{b\,\sigma = \mathit{true} \quad \mathrm{BS}\ c\ \sigma_1\ \sigma_2 \quad \mathrm{BS}\ (\textbf{while}\ b\ \textbf{do}\ c)\ \sigma_2\ \sigma_3}{\mathrm{BS}\ (\textbf{while}\ b\ \textbf{do}\ c)\ \sigma_1\ \sigma_3}$$

Step-Indexed Semantics

- Motivation: executable evaluation function for IMP
- Problem: Possible divergence of programs leads to divergence of the evaluation function
- Solution: decreasing index
  - guarantees termination
  - denotes the depth limit of the recursion tree
- SI : $\mathbb{N} \rightarrow \text{IMP} \rightarrow \Sigma \rightarrow \text{option } \Sigma$
- SI $n\ c\ \sigma = \lceil \tau \rceil$:= at recursion depth of at most n the execution of c in σ terminates in τ
- SI $n\ c\ \sigma = \bot$:= the execution of c in σ does not terminate in n steps
- Relation between Big-Step Semantics and Step-Indexed Semantics :

$$\text{BS } c\ \sigma\ \tau \Leftrightarrow \exists n.\ \text{SI } n\ c\ \sigma = \lceil \tau \rceil$$

Weakest Precondition Semantics

- Motivation: Observation of partial assignments instead of whole states
- Conditions = predicates on states: $\Sigma \to \mathit{Prop}$
- Does the execution of p in σ terminates in a state τ that satisfies $\lambda\tau.\tau\, x = 8$?
- Characterization by a predicate:
  WP $c\ \sigma\ Q$:= the execution of c in state σ terminates in a state that satisfies Q

Weakest Precondition Semantics

$$\frac{a\,\sigma = \tau \quad Q(\tau)}{\text{WP } a\ \sigma\ Q} \qquad \frac{\text{WP } c\ \sigma\ P \quad \text{WP } d\ P\ Q}{\text{WP } (c\ ;\ d)\ \sigma\ Q}$$

$$\frac{b\,\sigma = \mathit{true} \quad \text{WP } c\ \sigma\ Q}{\text{WP } (\textbf{if } b \textbf{ then } c \textbf{ else } d)\ \sigma\ Q} \qquad \frac{b\,\sigma = \mathit{false} \quad \text{WP } d\ \sigma\ Q}{\text{WP } (\textbf{if } b \textbf{ then } c \textbf{ else } d)\ \sigma\ Q}$$

$$\frac{b\,\sigma = \mathit{true} \quad \text{WP } c\ \sigma\ P \quad \text{WP } (\textbf{while } b \textbf{ do } c)\ P\ Q}{\text{WP } (\textbf{while } b \textbf{ do } c)\ \sigma\ Q}$$

$$\frac{b\,\sigma = \mathit{false} \quad Q\,(\sigma)}{\text{WP } (\textbf{while } b \textbf{ do } c)\ \sigma\ Q}$$

$$\text{WP } c\ P\ Q := \forall \sigma, P(\sigma) \rightarrow \text{WP } c\ \sigma\ Q$$

Weakest Preconditions

- Subsumption of all states σ satisfying WP c σ Q as the weakest precondition of c and Q:

$$\mathrm{wp}(c, Q) := \lambda\sigma.\mathrm{WP}\ c\ \sigma\ Q$$

- Definition of weakest preconditions via BS :

$$\mathrm{wp}_C(c, Q) := \lambda\sigma.\exists\tau, \mathrm{BS}\ c\ \sigma\ \tau \wedge Q\ \tau$$

- Coincidence of Big-Step and Weakest Precondition Semantics:

$$\mathrm{WP}\ c\ \sigma\ Q \leftrightarrow \mathrm{wp}_C(c, Q)\,\sigma$$
$$\mathrm{BS}\ c\ \sigma\ \tau \leftrightarrow \mathrm{WP}\ c\ \sigma\ (\lambda\tau'.\tau = \tau')$$

LIMP

- IMP not linear due to sequences and while-loops (needs a stack)
- Goal of compilation: Translation of IMP to a register transfer language
  - Sequences have to be linearized (no nesting)
  - While-loops have to be translated to blocks and calls

Example

while $b_{n>1}$ **do**

$a_{(n ::= n-1)}$;

$a_{(x ::= x \cdot x)}$

Example

block l : **if** $b_{n>1}$ **then** $a_{(n ::= n-1)}$; $(a_{(x ::= x \cdot x)}$; call $l)$

else halt ;

call l

LIMP

Alternative syntax for blocks and calls:

Example

$$\textbf{while } b_{n>1} \textbf{ do}$$
$$a_{(n ::= n-1)} \ ;$$
$$a_{(x ::= x \cdot x)}$$

Example

$$\textbf{fix } l.\, \textbf{if } b_{n>1} \textbf{ then } a_{(n ::= n-1)} \ ; \ a_{(x ::= x \cdot x)} \ ; \ l$$
$$\textbf{else halt}$$

$$s, t \ ::= \ \textbf{halt} \ \mid \ a \ ; \ s \ \mid \ \textbf{if } b \textbf{ then } c \textbf{ else } d \ \mid \ \textbf{fix } l.\, s \ \mid \ l = s \ ; \ t \ \mid \ l$$

Construct for non-recursive blocks helps to linearize conditionals (omittable)

LIMP

Example (p)

$$\begin{array}{l}\textbf{if } b_{x<0} \textbf{ then } a_{(x ::= -x)} \textbf{ else } a_{\textbf{SKIP}} \ ; \\ \textbf{while } b_{n>1} \textbf{ do} \\ \qquad a_{(n ::= n-1)} \ ; \\ \qquad a_{(x ::= x \cdot x)}\end{array}$$

Example

$$\begin{array}{l}k = \textbf{fix}\, l.\, \textbf{if } b_{n>1} \textbf{ then } a_{(n ::= n-1)} \ ; \ a_{(x ::= x \cdot x)} \ ; \ l \\ \qquad\qquad \textbf{else halt} \ ; \\ \textbf{if } b_{x<0} \textbf{ then } a_{(x ::= -x)} \ ; \ k \textbf{ else } k\end{array}$$

Weakest Precondition Semantics

$$\frac{Q(\sigma)}{\mathrm{WP}\ \mathbf{halt}\ \sigma\ Q} \qquad \frac{a\,\sigma = \tau \quad \mathrm{WP}\ s\ \sigma\ Q}{\mathrm{WP}\ (a\,;\,s)\ \sigma\ Q} \qquad \frac{b\,\sigma = \mathit{true} \quad \mathrm{WP}\ s\ \sigma\ Q}{\mathrm{WP}\ (\mathbf{if}\ b\ \mathbf{then}\ s\ \mathbf{else}\ t)\ \sigma\ Q}$$

$$\frac{b\,\sigma = \mathit{false} \quad \mathrm{WP}\ t\ \sigma\ Q}{\mathrm{WP}\ (\mathbf{if}\ b\ \mathbf{then}\ s\ \mathbf{else}\ t)\ \sigma\ Q} \qquad \frac{\mathrm{WP}\ s^{x}_{\mathbf{fix}\,x.\,s}\ \sigma\ Q}{\mathrm{WP}\ (\mathbf{fix}\,x.\,s)\ \sigma\ Q} \qquad \frac{\mathrm{WP}\ t^{x}_{s}\ \sigma\ Q}{\mathrm{WP}\ (x = s\,;\ t)\ \sigma\ Q}$$

- At most one recursive premise per rule
- No interpolants are needed
- Substitution semantics for fix and remember makes it unnecessary to keep track of introduced blocks

Compiler

- Problem: IMP-commands cannot be translated isolatedly as there is no sequence operator in LIMP to compose them

Example

$$\mathcal{C}\ (a_1) \rightsquigarrow a_1\ ;\ \textbf{halt} \qquad \mathcal{C}\ (a_2) \rightsquigarrow a_2\ ;\ \textbf{halt} \qquad \mathcal{C}\ (a_1\ ;\ a_2) \rightsquigarrow ?$$

- Solution: Translation of IMP-commands with respect to a continuation

$$\begin{aligned}
\mathcal{C}(_,_) &: \mathrm{IMP} \to \mathrm{LIMP} \to \mathrm{LIMP} \\
\mathcal{C}(a, s) &= a\ ;\ s \\
\mathcal{C}(c\ ;\ d, s) &= \mathcal{C}(c, \mathcal{C}(d, s)) \\
\mathcal{C}(\textbf{if}\ b\ \textbf{then}\ c\ \textbf{else}\ d, s) &= x = s\ ;\ \textbf{if}\ b\ \textbf{then}\ \mathcal{C}(c, x)\ \textbf{else}\ \mathcal{C}(d, x) && x\ \textit{fresh} \\
\mathcal{C}(\textbf{while}\ b\ \textbf{do}\ c, s) &= \textbf{fix}\, x.\ \textbf{if}\ b\ \textbf{then}\ \mathcal{C}(c, x)\ \textbf{else}\ s && x\ \textit{fresh}
\end{aligned}$$

Compiler in practice

- More convenient to use De Bruijn indices instead of names
- Makes it unnecessary to carry a counter as third argument

$$s, t \ ::= \ \textbf{halt} \ \mid \ a \,;\, s \ \mid \ \textbf{if } b \textbf{ then } c \textbf{ else } d \ \mid \ \textbf{fix } s \ \mid \ \textbf{rem } s \,;\, t \ \mid \ l$$

$$\begin{aligned}
\mathcal{C}(_,_) &: \mathsf{IMP} \to \mathsf{LIMP} \to \mathsf{LIMP} \\
\mathcal{C}(a, s) &= a \,;\, s \\
\mathcal{C}(c \,;\, d, s) &= \mathcal{C}(c, \mathcal{C}(d, s)) \\
\mathcal{C}(\textbf{if } b \textbf{ then } c \textbf{ else } d, s) &= \textbf{rem } s \,;\, \textbf{if } b \textbf{ then } \mathcal{C}(c, 0) \textbf{ else } \mathcal{C}(d, 0) \\
\mathcal{C}(\textbf{while } b \textbf{ do } c, s) &= \textbf{fix}\,\textbf{if } b \textbf{ then } \mathcal{C}(c, 0) \textbf{ else } s\uparrow^1
\end{aligned}$$

shift-operation $s\uparrow^1$ increases indices in s by 1 and prevents missreferencing

Compiler Correctness

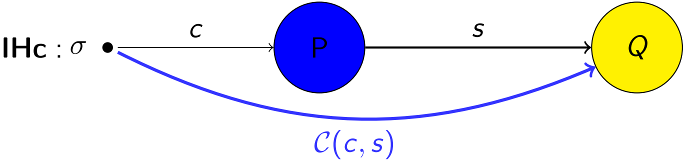
Goal: show correspondence of IMP-command c and LIMP-command $\mathcal{C}(c, \mathit{halt})$ with respect to weakest precondition semantics:

$$\text{WP } c\ \sigma\ Q \leftrightarrow \text{WP } \mathcal{C}(c, \mathit{halt})\ \sigma\ Q$$

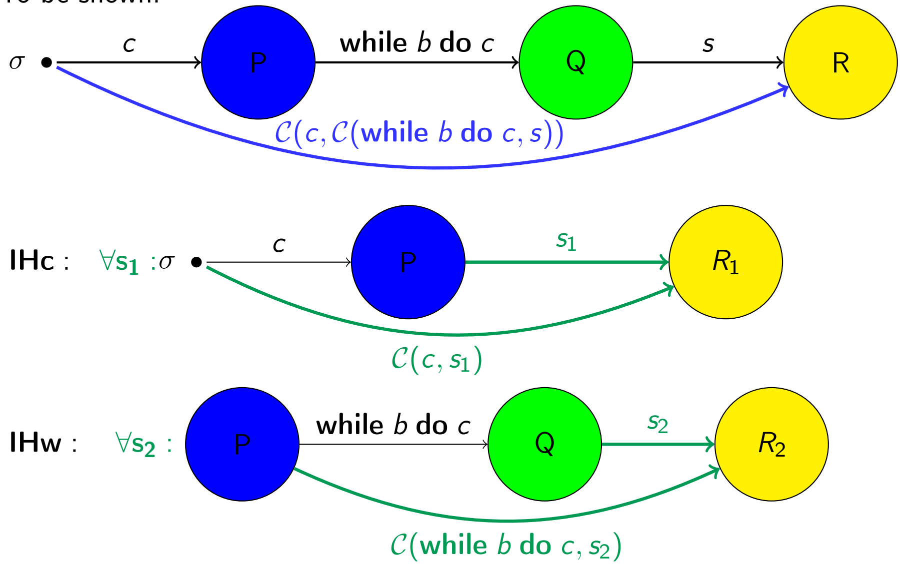
If-part:

- Generalization of the lemma for arbitrary continuations
- Effect of the continuation has to be taken into account
- Idea: $\mathcal{C}(c, s)$ first executes c and then continues with s

$$\text{WP } c\ \sigma\ P \rightarrow (\forall \sigma. P(\sigma) \rightarrow \text{WP } s\ \sigma\ Q) \rightarrow \text{WP } \mathcal{C}(c, s)\ \sigma\ Q$$



To be shown:



σ
c
P
while b do c
Q
s
R
$\mathcal{C}(c, \mathcal{C}(\textbf{while } b \textbf{ do } c, s))$
IHc :
$\forall s_1 : \sigma$
c
P
s_1
R_1
$\mathcal{C}(c, s_1)$
IHw :
$\forall s_2$:
P
while b do c
Q
s_2
R_2
$\mathcal{C}(\textbf{while } b \textbf{ do } c, s_2)$

Compiler Correctness

$$b\,\sigma = \mathit{true} \rightarrow$$
$$\mathrm{WP}\ (\mathcal{C}(c, \mathcal{C}(\textbf{while}\ b\ \textbf{do}\ c, s)))\ \sigma\ R \leftrightarrow \mathrm{WP}\ (\mathcal{C}(\textbf{while}\ b\ \textbf{do}\ c, s))\ \sigma\ R$$

Proof.

$$\begin{aligned}
& \mathrm{WP}\ (\mathcal{C}(\textbf{while}\ b\ \textbf{do}\ c, s))\ \sigma\ R \\
\overset{\mathit{Def.}\mathcal{C}}{\Longleftrightarrow}\ & \mathrm{WP}\ (\textbf{fix}\, x.\, \textbf{if}\ b\ \textbf{then}\ \mathcal{C}(c, x)\ \textbf{else}\ s)\ \sigma\ R \\
\overset{\mathit{subst.}}{\Longleftrightarrow}\ & \mathrm{WP}\ (\textbf{if}\ b\ \textbf{then}\ \mathcal{C}(c, \textbf{fix}\, x.\, \textbf{if}\ b\ \textbf{then}\ \mathcal{C}(c, x)\ \textbf{else}\ s)\ \textbf{else}\ s)\ \sigma\ R \\
\overset{b\,\sigma = \mathit{true}}{\Longleftrightarrow}\ & \mathrm{WP}\ (\mathcal{C}(c, \textbf{fix}\, x.\, \textbf{if}\ b\ \textbf{then}\ \mathcal{C}(c, s)\ \textbf{else}\ s))\ \sigma\ R \\
\overset{\mathit{Def.}\mathcal{C}}{\Longleftrightarrow}\ & \mathrm{WP}\ (\mathcal{C}(c, \mathcal{C}(\textbf{while}\ b\ \textbf{do}\ s, s)))\ \sigma\ R
\end{aligned}$$

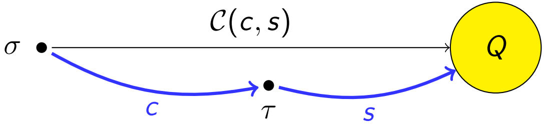
Substitution Lemma

$$\mathcal{C}(c, s)^x_t = \mathcal{C}(c, s^x_t)$$

Compiler Correctness

Idea: split up the execution of $\mathcal{C}(c, s)$ into the executions of c and s

$$\text{WP } \mathcal{C}(c, s)\ \sigma\ Q \to \exists \tau, \text{WP } c\ \sigma\ (\tau) \land \text{WP } s\ \tau\ Q$$



Proof by induction on c.

Problem: Inductive hypothesis in the while-true-case too weak.

Outlook

- Adding system calls to IMP and LIMP
  - Weakest Precondition Semantics with traces
  - Allows to compare non-terminating programs
- Weakest Precondition Semantics for functional IL

5 Appendix

Step-indexed Semantics

$$
\begin{aligned}
\text{SI} \quad &: \mathbb{N} \to \text{IMP} \to \Sigma \to \text{option } \sigma \\
\text{SI } 0\ c\ \sigma &= \bot \\
\text{SI } n\ (a)\ \sigma &= \lceil a\,\sigma \rceil \\
\text{SI } n\ (c\ ;\ d)\ \sigma &= \text{SI' } (n-1)\ d\ (\text{SI } (n-1)\ c\ \sigma) \\
\text{SI } n\ (\textbf{if } b \textbf{ then } c \textbf{ else } d)\ \sigma &= \textit{if } (b\,\sigma)\ \textit{then } \text{SI } (n-1)\ c\ \sigma\ \textit{else } \text{SI } (n-1)\ d\ \sigma \\
\text{SI } n\ (\textbf{while } b \textbf{ do } c)\ \sigma &= \textit{if } (b\,\sigma)\ \textit{then} \\
&\quad\ \text{SI' } (n-1)\ (\textbf{while } b \textbf{ do } c)\ (\text{SI } (n-1)\ c\ \sigma) \\
&\quad\ \textit{else } \lceil \sigma \rceil
\end{aligned}
$$

$$
\begin{aligned}
\text{SI'} \quad &: \mathbb{N} \to \text{IMP} \to \text{option } \Sigma \to \text{option } \Sigma \\
\text{SI' } n\ c\ \lceil \sigma \rceil &= \text{SI } n\ c\ \sigma \\
\text{SI' } n\ c\ \bot &= \bot
\end{aligned}
$$

Nested While-loop

Example

$$
\begin{aligned}
&\mathcal{C}(\textbf{while } b_1 \textbf{ do } (\textbf{while } b2 \textbf{ do } a), \textbf{halt}) \\
&= \textbf{fix}\,(\textbf{if } b_1 \textbf{ then } \mathcal{C}(\textbf{while } b_2 \textbf{ do } a, 0) \textbf{ else halt} \uparrow^1) \\
&= \textbf{fix}\,(\textbf{if } b_1 \textbf{ then fix}\,(\textbf{if } b_2 \textbf{ then } \mathcal{C}(a, 0) \textbf{ else } 0 \uparrow^1) \textbf{ else halt}) \\
&= \textbf{fix}\,(\textbf{if } b_1 \textbf{ then fix}\,(\textbf{if } b_2 \textbf{ then } a\,;\,0 \textbf{ else } 1) \textbf{ else halt})
\end{aligned}
$$

Compiler Correctness

- Idea: split up the execution of $\mathcal{C}(c, s)$ in the execution of c and s
- τ denotes the state after the execution of c in σ

$$\text{WP } \mathcal{C}(c, s)\ \sigma\ Q \rightarrow \exists \tau, \text{WP } c\ \sigma\ (\tau) \wedge \text{WP } s\ \tau\ Q$$

- we try to prove by induction on c, but fail in the while-case:

$$\begin{aligned}
\textbf{IHc} &: \text{WP } \mathcal{C}(c, s)\ \sigma\ Q \rightarrow \exists \tau, \text{WP } c\ \sigma\ (\tau) \wedge \text{WP } s\ \tau\ Q \\
\textbf{A} &: \text{WP } (\textbf{fix}\, x.\, \textbf{if } b \textbf{ then } \mathcal{C}(c, x) \textbf{ else } s)\ \sigma\ Q \\
&\overset{\textit{subst.}}{\Longrightarrow} \text{WP } (\textbf{if } b \textbf{ then } \mathcal{C}(c, \mathcal{C}(\textbf{while } b \textbf{ do } c, s)) \textbf{ else } s)\ \sigma\ Q \\
&\overset{\textit{IHc}}{\Longrightarrow} \exists \tau, \text{WP } c\ \sigma\ \tau \wedge \text{WP } \mathcal{C}(\textbf{while } c \textbf{ do } d, s)\ \tau\ Q \\
&\lightning\ \textit{Inductive hypothesis for while would be needed}
\end{aligned}$$

- Solution: Introduction a new step-indexed predicate for WP that keeps track of the number of substitutions in the fix-case
- Nested induction n the substitution depth in the while-case

Compiler Correctness

$$\text{WP}\ c\ \sigma\ P \rightarrow (\forall \sigma . P(\sigma) \rightarrow \text{WP}\ s\ \sigma\ Q) \rightarrow \text{WP}\ \mathcal{C}(c, s)\ \sigma\ Q$$

Proof by induction on WP $c\ \sigma\ P$:

Proof.

Case: **while** b **do** c (true):

$\mathbf{A_1}$: WP $c\ \sigma\ P$ $\quad$ $\mathbf{A_2}$: WP (**while** b **do** c) $P\ Q$ $\quad$ $\mathbf{A_3}$: WP $s\ Q\ R$

$$\mathbf{IHc} : \text{WP}\ s'\ P\ Q' \rightarrow \text{WP}\ (\mathcal{C}(c, s'))\ \sigma\ Q'$$

$$\mathbf{IHw} : \text{WP}\ s'\ Q\ Q' \rightarrow \text{WP}\ (\mathcal{C}(\textbf{while}\ b\ \textbf{do}\ c, s'))\ P\ Q'$$

$$\mathbf{A_3} : \text{WP}\ s\ Q\ R$$

$$\stackrel{IHw}{\Longrightarrow} \text{WP}\ (\mathcal{C}(\textbf{while}\ b\ \textbf{do}\ c, s))\ P\ R$$

$$\stackrel{IHc}{\Longrightarrow} \text{WP}\ (\mathcal{C}(c, \mathcal{C}(\textbf{while}\ b\ \textbf{do}\ c, s)))\ \sigma\ R$$

$\square$